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Generalized Uncertainty Principle Influences the Entropy of a Nonstationary Black Hole

Chengzhou Liu,[1,2,4] Xiang Li,[3] and Zheng Zhao[1]

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The entropy of a scalar field at the horizon is investigated in the Vaidya space–time. We take into account the effect of the generalized uncertainty principle on the state density and the entropy. The divergence in the brick-wall model is removed and the entropy proportional to the horizon area is obtained.

KEY WORDS: generalized uncertainty principle; black hole; horizon; enthropy.

The discovery of the Hawking radiation (Hawking, 1975) confirms the conjecture that the black-hole entropy is proportional to the horizon area (Bekenstein, 1973). Many efforts have been devoted to the explanation of the black-hole entropy (Gao and Liu, 2000; Hod, 2000; Jing, 1998; Li, 2000; Li and Zhao, 2000; Li and Zhao, 2001; Liberati *et al.*, 1997; Liu and Zhao, 2000; Liu and Zhao, 2001; Mukohyama and Israel, 1998; Racz, 2000; Shen, 1997; Teitelboim, 1995; ‘t Hooft, 1985). The brick-wall model proposed by ‘t Hooft (1985), tries to attribute the black-hole entropy to the quantum states of the field outside the horizon, in which the entropy proportional to the horizon area is obtained. But it is dependent on a cutoff near the horizon. The later investigation argue that the black-hole entropy derives from the “wall contribution” (Mukohyama and Israel, 1998), that is, because of the degree of freedom of the field near the horizon (Gao and Liu, 2001; Li and Zhao, 2001; Liu and Zhao, 2001). Thus we can compute the entropy of a nonstationary black hole in the context of the local equilibrium in the vicinity of the horizon (Li and Zhao, 2001). The entropy is still divergent as the cutoff vanishes. However, it is pointed out that (Li, 2002) the generalized uncertainty principle (GUP) essentially influences the density of the states near the horizon and removes the divergence appearing in the brick-wall model. Here, we apply the

[1] Department of Physics, Beijing Normal University, Beijing, People’s Republic of China.
[2] Department of Physics, Binzhou Teachers College, Binzhou, Shandong, People’s Republic of China.
[3] Institute of Theoretical Physics, Chinese Academy of Science, Beijing, People’s Republic of China.
[4] To whom correspondence should be addressed at Department of Physics, Beijing Normal University, Beijing 100875, People’s Republic of China. E-mail: iczbzbj2003@sina.com

GUP to a nonstationary case: Vaidya black hole. We follow the preceding work (Li and Zhao, 2001), the entropy of a nonstationary hole is investigated by the membrane approach: the entropy is attributed to the field on the membrane which is a 2D surface just at the horizon. This is based on the following reasons: the entropy of the hole is related to the existence of the horizon, that is, the entropy is the intrinsic property of the horizon. Furthermore, as an extensive quantity, the entropy should be proportional to the volume (for a 3D system) or the area (for a 2D system). Therefore, it is natural to count the quantum number of the horizon. Unfortunately, the cutoff introduced by hand is necessary to regulate the divergence of the entropy in the preceding work (Li and Zhao, 2001). This difficulty can be overcome by the GUP.

Let us start with the geometry of the Vaidya black hole (Carmeli, 1982)

$$ds^2 = -\left(1 - \frac{2m(\upsilon)}{r}\right) d\upsilon^2 + 2\, d\upsilon\, dr + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2, \tag{1}$$

which describes an evoporating hole with mass loss rate $\dot{m} = \partial m/\partial\upsilon$.

The location of the horizon is given by (Luo and Zhao, 1993)

$$r_h = \frac{2m}{1 - 2\dot{r}_h}, \tag{2}$$

where $\dot{r}_h = \partial r_h/\partial\upsilon$. Obviously, the horizon does not coincide with the infinite redshift-surface $r = 2m(\upsilon)$. However, this is coordinate-dependent. Introducing the following transformation (Li, 1999):

$$R = r - r_h, \qquad dR = dr - \dot{r}_h\, d\upsilon, \tag{3}$$

Eq. (1) can be reduced to

$$ds^2 = -\left(1 - 2\dot{r}_h - \frac{2m}{r}\right) d\upsilon^2 + 2\, d\upsilon\, dR + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4}$$

The horizon is directly located by $g_{00} = 0$. We let $dR = 0$ and obtain

$$ds_m^2 = -\left(1 - 2\dot{r}_h - \frac{2m}{r}\right) d\upsilon^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{5}$$

which describes a surface outside the horizon. Substituting (5) into the massless scalar field equation as follows:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0, \tag{6}$$

we obtain

$$-\frac{r^2}{a}\partial_\upsilon^2\Phi - \frac{1}{\sqrt{a}}\partial_\upsilon(r^2/\sqrt{a})\partial_\upsilon\Phi + ctg\theta\,\partial_\theta\Phi + \partial_\theta^2\Phi + \frac{1}{\sin^2\theta}\partial_\varphi^2\Phi = 0, \tag{7}$$

where

$$a = 1 - 2\dot{r}_h - \frac{2m}{r}. \tag{8}$$

We substitute the ansatz

$$\Phi = F(\upsilon, r)Y_{lm}(\theta, \varphi), \tag{9}$$

into (7) and obtain

$$r^2 \frac{\partial^2 F}{\partial \upsilon^2} + \sqrt{a}\partial_\upsilon(r^2/\sqrt{a})\frac{\partial F}{\partial \upsilon} - ab^2 F = 0 \tag{10}$$

and

$$ctg\theta \frac{\partial Y}{\partial \theta} + \frac{\partial^2 Y}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2} + b^2 Y = 0, \tag{11}$$

where b is the separation constant. Setting

$$Y = e^{iZ(\theta,\varphi)}, \tag{12}$$

we obtain

$$\left(\frac{\partial Z}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial Z}{\partial \varphi}\right)^2 - b^2 = 0. \tag{13}$$

The momentums respectively conjugate to θ, φ read

$$p_\theta = \frac{\partial z}{\partial \theta}, \qquad p_\varphi = \frac{\partial Z}{\partial \varphi}; \tag{14}$$

thus we obtain the square modulus of momentum

$$P^2 = P_i P^i = g_m^{11} P_\theta^2 + g_m^{22} P_\varphi^2 = \frac{b^2}{r_h^2}. \tag{15}$$

Because of the GUP (Adler *et al.*, 2001; Ahluwalia, 2000; Chang *et al.*, 2002; Garay, 1995; Kastrup, 1997; Kempt *et al.*, 1995; Rama, 2001)

$$\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar}(\Delta p)^2, \tag{16}$$

the equation of the state density should be modified by

$$dn = \frac{d^2\vec{x}\, d^2\vec{p}}{(2\pi\hbar)^2(1+\lambda p^2)^2}, \tag{17}$$

where $\lambda \sim G$ is of order of the Planck area l_p^2. Note that the state density of the high frequency is suppressed by a factor $(1+\lambda p^2)^{-2}$. However, (17) returns to the usual case if $\lambda \to 0$. In the following discussion, we take the natural units $\hbar = c = G = k_B = 1$.

From (17), the number of quantum states with energy less than ω is given by

$$\Gamma(\omega) = \int dn = \frac{1}{(2\pi)^2 \left(1 + \lambda \frac{b^2}{r_h^2}\right)^2} \int d\theta\, d\varphi\, dp_\theta\, dp_\varphi, \tag{18}$$

where b is to be determined. (10) has the form of the standard wave equation, in terms of the tortoise coordinate (Zhao and Dai, 1992); this means $F \sim e^{-i\omega\upsilon}$. Generally, we let

$$F = f(\upsilon, r)e^{-i\omega\upsilon}; \tag{19}$$

then (10) can be reduced to

$$r^2(\ddot{f} - \omega^2 f) + \sqrt{a}\,\dot{f}\,\overline{\partial}\upsilon(r^2/\sqrt{a}) + ab^2 f = 0, \tag{20}$$

$$2r^2\dot{f} + \sqrt{a}\overline{\partial}\upsilon(r^2/\sqrt{a}) f = 0, \tag{21}$$

Where $\dot{f} = \partial f/\partial\upsilon$. Substituting (21) into (20), we have

$$b^2 = \frac{r^2}{a}\left(\omega^2 + \frac{2\dot{f}^2}{f^2} - \frac{\ddot{f}}{f}\right) = \frac{r^2}{a}(\omega^2 + \Lambda), \tag{22}$$

where

$$\Lambda = \frac{2\dot{f}^2}{f^2} - \frac{\ddot{f}}{f}. \tag{23}$$

Now we study the asymptotic form of $\dot{f}(\upsilon, r)$ at the horizon. From (20), we have

$$ar^2(\ddot{f} - \omega^2 f) + \dot{f}\left(2ar\dot{r} - \frac{1}{2}r^2\dot{a}\right) + a^2b^2 f = 0, \tag{24}$$

where $\dot{r} = \partial r/\partial\upsilon$, $\dot{a} = \partial a/\partial\upsilon$. At the horizon, we have

$$r = r_h, \qquad a = 0, \tag{25}$$

then

$$\dot{f}(r_h, \upsilon) = 0, \tag{26}$$

which means that f is independent of υ at the horizon. Near the horizon, the solution to (10) reads

$$F \sim e^{-i\omega\upsilon}, \tag{27}$$

and

$$\Lambda = 0, \qquad b^2 = r^2\omega^2/a. \tag{28}$$

Because of (15) and (22), (18) is reduced to

$$\Gamma(\omega) = \frac{2}{(2\pi)^2\left(1+\lambda\frac{b^2}{r^2}\right)^2}\int d\theta\, d\varphi \int \sqrt{b^2 - p_\theta^2}\,\sin\theta\, dp_\theta = \frac{b^2}{\left(1+\lambda\frac{b^2}{r^2}\right)^2}, \tag{29}$$

where the integration goes over those values of p_θ for which the argument of the square root is positive. According to quantum statistical mechanics, the free energy reads

$$F = -\int_0^\infty \frac{\Gamma(\omega)}{e^{\beta\omega}-1}\,d\omega = -\int_0^\infty \frac{b^2\,d\omega}{(e^{\beta\omega}-1)\left(1+\lambda\frac{b^2}{r^2}\right)^2}, \tag{30}$$

and the entropy

$$\begin{aligned} S &= \beta^2\frac{\partial F}{\partial\beta} = \beta^2\int_0^\infty \frac{\omega\, e^{\beta\omega} b^2}{(e^{\beta\omega}-1)^2\left(1+\lambda\frac{b^2}{r^2}\right)^2}\,d\omega \\ &= \frac{r^2\beta^2}{a}\int_0^\infty \frac{(\omega^2+\Lambda)\omega\, d\omega}{(1-e^{-\beta\omega})(e^{\beta\omega}-1)\left(1+\frac{\lambda(\omega^2+\Lambda)}{a}\right)^2}. \end{aligned} \tag{31}$$

Note that $\Lambda = 0$ at the horizon. We are only interested in the contribution from the quantum states covering the horizon, then

$$S_0 = \frac{r^2\beta^2}{a}\int_0^\infty \frac{\omega^3\, d\omega}{(1-e^{-\beta\omega})(e^{\beta\omega}-1)\left(1+\frac{\lambda}{a}\omega^2\right)^2}. \tag{32}$$

By using two inequalities

$$\begin{aligned} e^{\beta\omega} - 1 &> \beta\omega, \\ 1 - e^{-\beta\omega} &> \frac{\beta\omega}{1+\beta\omega}, \end{aligned} \tag{33}$$

we obtain the upper bound of the entropy

$$\begin{aligned} S_0 &< \frac{r^2}{a}\int_0^\infty \frac{\omega+\beta\omega^2}{\left(1+\frac{\lambda}{a}\omega^2\right)^2}\,d\omega = \frac{r^2}{a}\left[\frac{a}{2\lambda} + \frac{\pi\beta}{4}\left(\frac{a}{\lambda}\right)^3\right] \\ &= r^2\left(\frac{1}{2\lambda} + \frac{\pi\beta a^{1/2}}{4\lambda^{3/2}}\right) = \frac{A_h}{8\pi\lambda}. \end{aligned} \tag{34}$$

At the horizon $a = 0$, $A_h = 4\pi r_h^2$.

So we obtain the entropy bound of a Vaidya hole by counting the quantum states of the field covering the horizon. It is still proportional to the horizon area. Comparing to the ‘t Hooft brick-wall model (‘t Hooft, 1985), the entropy, as shown by (34), is convergent, without any cutoff. Note that the entropy of a Vaidya hole is less than a static hole that with same mass. This can be understood as follows: a Vaidya hole is in a nonequilibrium state with less entropy than the equilibrium state. This work is supported by the National Science Foundation of China under Grant No. 10073002.

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